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Renormalized set of equations for the Green functions in the Yang–Mills field theory and its asymptotic solution

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Abstract. The set of completely renormalized equations which is free of difficulties connected with the treatment of so called overlapping divergencies is obtained in the Yang–Mills field theory. The asymptotically strict solution of this set is found in the region of large transferred momentum. The ultraviolet asymptotic behaviour of all the Green and vertex functions is obtained and the connection between the bare and the experimental charge is discussed.

1. Introduction

Recent investigations by several authors (Gross and Wilczek 1973a, b, Politzer 1973, Cheng *et al* 1974) have shown that there exists a certain class of field theories free of the known 'zero charge' difficulty (Landau and Pomeranchuk 1955, Fradkin 1955a). The gauge Yang–Mills fields which play the important role in the unified theory of weak, electromagnetic and strong interactions belong to this class of theories. This remarkable fact was established by Gross and Wilczek (1973) and by Politzer (1973) by summing the perturbation series on the basis of the renormalization group method. The absence of the 'zero charge' difficulty from this theory indicates its logical self-consistency and is of greatest importance for further study. At present it seems doubtless that the Yang–Mills theory will be intrinsically involved in a wide range of problems of future quantum field theory. In this connection we think it reasonable to formulate the Yang–Mills field theory in such a closed form as would admit studying by the methods which proceed far beyond the scope of ordinary perturbation theory.

In this paper a set of completely renormalized equations will be obtained for the Green functions and the asymptotically strict solution will be found in the large-momentum transfer limit (Fradkin and Kalashnikov 1974). We shall proceed from the set of dynamical equations describing this class of theories and reformulate them into the set of unrenormalized equations for the Green functions. The set of completely renormalized equations which is free of the known difficulty of treating overlapping divergencies is then obtained following the method of one of the authors (Fradkin 1954, 1955b, 1965). In conclusion this set of equations is solved within the 'three-gamma' approximation which provides an asymptotically precise result for all the Green and vertex functions. The connection between the 'bare' and experimental charges is established and the absence of the 'zero charge' difficulty for this class of theories is discussed.

There is also another reason why we think it important to carry out the above-mentioned programme in the framework of dynamical equations. The point is that in our opinion a number of problems in modern gauge field theory (such as the possibility that the quark may form bound states when interacting through the Yang–Mills fields or, say, temperature and many-particle effects in various models of the unified interaction, etc) can be solved consistently only when a corresponding set of renormalized dynamical equations is employed. In this connection the set of completely renormalized equations for the Green functions in the Yang–Mills field theory obtained in the present paper may turn out to be useful for a number of other applications as well.

2. A set of unrenormalized equations for the Green functions

We shall proceed from the well known expression for the generating functional (De Witt 1967, Fadeev and Popov 1967, Mandelstam 1968, Fradkin and Tyutin 1970) of the Yang–Mills field theory

$$\mathscr{Z} = \int \mathscr{D}[\ldots]\exp(\mathrm{i}S),$$
$$S[\ldots] = -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \bar{C}^a(\nabla^{ab}_\mu\partial_\mu)C^b + \tfrac{1}{2}\alpha(\partial_\mu A^a_\mu)^2 + A^a_\mu J^a_\mu + [\bar{C}^a\eta^a + \bar{\eta}^a C^a] \tag{2.1}$$

by transforming it identically to more convenient variables

$$\begin{aligned} S[\ldots] = {} & -\frac{1}{2}A(1)[D_{A^2}]_0^{-1}(1;2)A(2) + \frac{\mathrm{i}g_0}{3!}\Gamma^{(0)}_{A^3}(1;2;3)A(1)A(2)A(3) \\ & + \frac{(\mathrm{i}g_0)^2}{4!}\Gamma^{(0)}_{A^4}(1;2;3;4)A(1)A(2)A(3)A(4) - \bar{C}(1)[G_{C\bar{C}}]_0^{-1}(1;2)C(2) \\ & + (\mathrm{i}g_0)\Gamma^{(0)}_{C\bar{C}A}(1;2|3)\bar{C}(1)C(2)A(3) + A(1)J(1) + [\bar{C}(1)\eta(1) + \bar{\eta}(1)C(1)]. \end{aligned} \tag{2.2}$$

Summation and integration over the repeated labels is implied here. The zero Green functions

$$\begin{aligned} ([D_{A^2}]_0^{-1})^{ab}_{\mu\nu}(x;y) &= \delta(x-y)\delta^{ab}[-\delta_{\mu\nu}\Box + (1+\alpha)\partial_\mu\partial_\nu], \\ ([G_{C\bar{C}}]_0^{-1})^{ab}(x;y) &= \delta(x-y)\delta^{ab}(-\Box) \end{aligned} \tag{2.3}$$

and the bare vertex functions

$$\begin{aligned} (\Gamma^{(0)}_{A^3})^{abc}_{\eta\gamma\sigma}&(x;y;z) \\ = {} & \mathrm{i}f^{abc}\left\{\delta_{\eta\gamma}\left[-2\delta(x-z)\left(\frac{\partial}{\partial y_\sigma}\delta(y-x)\right) + \delta(x-y)\left(\frac{\partial}{\partial x_\sigma}\delta(x-z)\right)\right]\right. \\ & + \delta_{\eta\sigma}\left[-2\delta(x-y)\left(\frac{\partial}{\partial x_\gamma}\delta(x-z)\right) + \delta(x-z)\left(\frac{\partial}{\partial y_\gamma}\delta(x-y)\right)\right] \\ & \left. + \delta_{\gamma\sigma}\left[\delta(x-z)\left(\frac{\partial}{\partial y_\eta}\delta(y-x)\right) + \delta(x-y)\left(\frac{\partial}{\partial x_\eta}\delta(x-z)\right)\right]\right\}, \end{aligned} \tag{2.4}$$

$$(\Gamma^{(0)}_{C\bar{C}A})^{ab}|^c_\mu(x;y|z) = +\mathrm{i}f^{abc}\left(\frac{\partial}{\partial z_\mu}\delta(z-y)\right)\delta(x-z),$$

$$\Gamma^{(0)abcd}_{A^4\,\mu\nu\sigma\delta}(z_1;z_2;z_3;z_4) = \delta(z_1-z_2)\delta(z_1-z_3)\delta(z_1-z_4)\{(f^{pab}\cdot f^{pcd})[\delta_{\delta\mu}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\delta}] + (f^{pbc}\cdot f^{pad})[\delta_{\delta\sigma}\delta_{\mu\nu}-\delta_{\nu\sigma}\delta_{\mu\delta}] + (f^{pac}\cdot f^{pbd})[\delta_{\delta\sigma}\delta_{\mu\nu}-\delta_{\nu\delta}\delta_{\mu\sigma}]\}$$

are obtained from the initial functional $S[\ldots]$. The exact Green functions

$$D_{A^2}(1;2) = -\mathrm{i}\frac{\delta^2 \ln \mathscr{Z}}{\delta J(1)\delta J(2)}, \qquad G_{C\bar{C}}(1;2) = -\mathrm{i}\frac{\delta^2 \ln \mathscr{Z}}{\delta\bar{\eta}(1)\delta\eta(2)} \tag{2.5}$$

and the vertex function of the theory under consideration

$$\begin{aligned} \Gamma_{A^3}(1;2;3) &= -\frac{\delta D_{A^2}^{-1}(1;2)}{\mathrm{i}g_0\delta\langle A(3)\rangle}, \\ \Gamma_{A^4}(1;2;3;4) &= -\frac{\delta^2 D_{A^2}^{-1}(1;2)}{(\mathrm{i}g_0)^2\delta\langle A(3)\rangle\delta\langle A(4)\rangle}, \\ \Gamma_{C\bar{C}A}(1;2|3) &= -\frac{\delta G_{C\bar{C}}^{-1}(1;2)}{\mathrm{i}g_0\delta\langle A(3)\rangle} \end{aligned} \tag{2.6}$$

are defined here in a usual way.

The set of functional equations is derived by a usual method and, according to (2.2), has the following simple form:

$$\begin{aligned} &[D_{A^2}]_0^{-1}(1;2)\langle A(2)\rangle - \frac{g_0}{2}\Gamma^{(0)}_{A^3}(1;2;3)\left[D_{A^2}(2;3)-\frac{1}{\mathrm{i}}\langle A(2)\rangle\langle A(3)\rangle\right] \\ &\quad + \frac{(\mathrm{i}g_0)^2}{6}\Gamma^{(0)}_{A^4}(1;2;3;4)\left[\frac{\delta D_{A^2}(2;3)}{\delta J(4)}-\frac{1}{\mathrm{i}}\langle A(2)\rangle D_{A^2}(3;4)\right. \\ &\quad \left.-\frac{1}{\mathrm{i}}\langle A(3)\rangle D_{A^2}(2;4)-\frac{1}{\mathrm{i}}\langle A(4)\rangle D_{A^2}(2;3)-\langle A(2)\rangle\langle A(3)\rangle\langle A(4)\rangle\right] \\ &\quad + g_0\Gamma^{(0)}_{AC\bar{C}}(1|2;3)\left[G_{C\bar{C}}(3;2)-\frac{1}{\mathrm{i}}\langle C(3)\rangle\langle\bar{C}(2)\rangle\right] = J(1), \\ &[G_{C\bar{C}}]_0^{-1}(1;2)\langle C(2)\rangle - g_0\Gamma^{(0)}_{C\bar{C}A}(1;2|3)\left[\frac{\delta\langle C(2)\rangle}{\delta J(3)}-\frac{1}{\mathrm{i}}\langle C(2)\rangle\langle A(3)\rangle\right] = \eta(1). \end{aligned} \tag{2.7}$$

The set of unrenormalized equations for the corresponding Green functions is then obtained from (2.7) by differentiation of the latter with respect to the external sources. A simple algebra, it may be easily put into the form of the Schwinger–Dyson-type equations:

$$\begin{aligned} [D_{A^2}]^{-1}(1;2) &= [D_{A^2}]_0^{-1}(1;2)-(\mathrm{i}g_0)\Gamma^{(0)}_{A^3}(1;2;3)\langle A(3)\rangle \\ &\quad -\tfrac{1}{2}(\mathrm{i}g_0)^2\Gamma^{(0)}_{A^4}(1;2;3;4)\langle A(3)\rangle\langle A(4)\rangle-\Pi_{A^2}(1;2), \\ [G_{C\bar{C}}]^{-1}(1;2) &= [G_{C\bar{C}}]_0^{-1}(1;2)-\Sigma_{C\bar{C}}(1;2) \end{aligned} \tag{2.8}$$

The sources of the C-field are switched off here and the self-energy operators are defined by the following integral representations:

$$\Pi_{A^2}(1;\bar{1}) = -\frac{g_0^2}{2i}\Gamma^{(0)}_{A^4}(1;2;3;\bar{1})D_{A^2}(2;3) - \frac{g_0^2}{2i}\Gamma^{(0)}_{A^3}(1;2;3)D_{A^2}(2;\bar{3})\Gamma_{A^3}(\bar{3};\bar{2};\bar{1})D_{A^2}(\bar{2};3)$$
$$-\frac{g_0^3}{2}\Gamma^{(0)}_{A^4}(1;2;3;4)\langle A(2)\rangle D_{A^2}(4;\bar{3})\Gamma_{A^3}(\bar{3};\bar{2};\bar{1})D_{A^2}(\bar{2};3)$$
$$-\frac{g_0^4}{2}\Gamma^{(0)}_{A^4}(1;2;3;4)D_{A^2}(2;\bar{4})\Gamma_{A^3}(\bar{4};5;6)$$
$$\times D_{A^2}(5;\bar{2})\Gamma_{A^3}(\bar{3};\bar{2};\bar{1})D_{A^2}(\bar{3};4)D_{A^2}(6;3) \tag{2.9}$$
$$-\frac{g_0^4}{6}\Gamma^{(0)}_{A^4}(1;2;3;4)D_{A^2}(2;\bar{2})\Gamma_{A^4}(\bar{4};\bar{3};\bar{2};\bar{1})D_{A^2}(3;\bar{3})D_{A^2}(\bar{4};4)$$
$$+\frac{g_0^2}{i}\Gamma^{(0)}_{AC\bar{C}}(1|2;3)G_{C\bar{C}}(3;\bar{3})\Gamma_{C\bar{C}A}(\bar{3};\bar{2}|\bar{1})G_{C\bar{C}}(\bar{2};2),$$

$$\Sigma_{C\bar{C}}(1;\bar{1}) = -\frac{g_0^2}{i}\Gamma^{(0)}_{C\bar{C}A}(1;2|3)G_{C\bar{C}}(2;\bar{2})\Gamma_{AC\bar{C}}(\bar{3}|\bar{2};\bar{1})D_{A^2}(\bar{3};3).$$

All the necessary vertex functions are to be found by direct differentiation of (2.9) according to their definition (2.6).

3. Renormalization of the set of equations for the Green function

Although the set of equations (2.8) is closed and consistent, explicit calculations with the aid of it turn out to be extremely difficult. The point is that all the quantities involved appear to be infinite due to divergency of the corresponding integrals in the region of both small and large momenta. The divergencies in the region of small momenta are due to the masslessness of the Yang–Mills fields which introduce an actual difficulty to the theory in question. On the contrary, the ultraviolet divergencies are easily eliminated from the theory within a usual renormalization programme. The latter can be carried out in a general form directly in the framework of the set of equations (2.8).

For that purpose it is necessary first of all to introduce the Z-factors to the Green and vertex functions

$$\Gamma^R = Z_1\cdot\Gamma, \qquad G^R = Z_2^{-1}\cdot G, \qquad D^R = Z_3^{-1}\cdot D \tag{3.1}$$

and to go over in (2.8) to the renormalized quantities

$$[D^R_{A^2}]^{-1}(1;2) = Z_3^{A^2}\cdot[D_{A^2}]_0^{-1}(1;2) - (ig)Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}(1;2;3)\langle A^R(3)\rangle$$
$$-\frac{(ig)^2}{2}\cdot Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}(1;2;3;4)\langle A^R(3)\rangle\langle A^R(4)\rangle - \Pi'_{A^2}(1;2), \tag{3.2}$$
$$[G^R_{C\bar{C}}]^{-1}(1;2) = Z_2^{C\bar{C}}\cdot[G_{C\bar{C}}]_0^{-1}(1;2) - \Sigma'_{C\bar{C}}(1;2),$$

with account taken of the exact Ward identities (Taylor 1971, Slavnov 1972)

$$Z_1^{A^3}\cdot[Z_3^{A^2}]^{-1} = Z_1^{AC\bar{C}}\cdot[Z_2^{C\bar{C}}]^{-1}, \qquad Z_1^{A^4} = [Z_1^{A^3}]^2\cdot[Z_3^{A^2}]^{-1} \tag{3.3}$$

as well as the connection between the 'bare' and experimental charges

$$g^2 = g_0^2 \cdot [Z_1^{A^3}]^{-2} \cdot [Z_3^{A^2}]^3. \tag{3.4}$$

Then in view of the arbitrariness of the theory one can choose the Z-factors so that all the ultraviolet divergencies of the theory be eliminated. In particular, the Z_2- and Z_3-factors are determined from the requirement that the corresponding divergencies in the Green functions be eliminated:

$$Z_2^{C\bar{C}} = 1 + \frac{\partial \Sigma'_{C\bar{C}}(k_0^2)}{\partial k_0^2}, \qquad Z_3^{A^2} = 1 + \frac{\partial \Pi'_{A^2}(k_0^2)}{\partial k_0^2} \tag{3.5}$$

while the Z_1-factors serve for the elimination of ultraviolet divergencies in the vertex functions. The renormalized Green functions retain the structure of the Schwinger–Dyson-type equations

$$\begin{aligned}
[D_{A^2}^R]^{-1}(1;2) &= [D_{A^2}]_0^{-1}(1;2) - (ig)Z_1^{A^3} \cdot \Gamma_{A^3}^{(0)}(1;2;3)\langle A^R(3)\rangle \\
&\quad - \tfrac{1}{2}(ig)^2 \cdot Z_1^{A^4} \cdot \Gamma_{A^4}^{(0)}(1;2;3;4)\langle A^R(3)\rangle\langle A^R(4)\rangle - \Pi_{A^2}^R(1;2), \\
[G_{C\bar{C}}^R]^{-1}(1;2) &= [G_{C\bar{C}}]_0^{-1}(1;2) - \Sigma_{C\bar{C}}^R(1;2),
\end{aligned} \tag{3.6}$$

however, self-energy operators in these equations

$$\begin{aligned}
\tilde{\Pi}_{A^2}^R(k^2) &= \tilde{\Pi}'_{A^2} - k^2 \frac{\partial \tilde{\Pi}'(k_0^2)}{\partial k_0^2} \\
\tilde{\Sigma}_{C\bar{C}}^R(k^2) &= \tilde{\Sigma}'_{C\bar{C}} - k^2 \frac{\partial \tilde{\Sigma}'(k_0^2)}{\partial k_0^2}
\end{aligned} \tag{3.7}$$

do not contain ultraviolet divergencies. Here k_0^2 is an arbitrary point of renormalization, $\tilde{\Pi}'_{A^2}$ and $\tilde{\Sigma}'_{C\bar{C}}$ are those parts of the self-energy operators which are subject to renormalization. They have a tensor structure similar to that of the corresponding lowest-order Green function.

At the next stage the 'overlapping' divergencies connected with the resolution of $0\cdot\infty$-type uncertainty should be eliminated from the theory. This uncertainty arises when one solves the set of renormalized equations (3.6) due to the presence in it of the Z-factors and 'bare' vertices. We shall apply here the same method as the one applied in the papers (Fradkin 1954, 1955b, 1965) by one of the present authors, and after some easy transformations we shall effectively eliminate the above-mentioned difficulty. Thus a completely renormalized set of equations for the Green functions in the Yang–Mills field theory is obtained.

We begin with eliminating the $(Z_1^{AC\bar{C}} \cdot \Gamma_{AC\bar{C}}^{(0)})$ combination. For that purpose, instead of the four-point function $W_{C\bar{C}A^2}(1;2|3;4)$,

$$\begin{aligned}
W_{C\bar{C}A^2}&(1;2|3;4) \\
&= \frac{\delta \Gamma_{C\bar{C}A}^R(1;2|4)}{(ig)\delta\langle A^R(3)\rangle} + \Gamma_{C\bar{C}A}^R(1;\bar{2}|3)G_{C\bar{C}}^R(\bar{2};\bar{1})\Gamma_{C\bar{C}A}^R(\bar{1};2|4) \\
&\quad + \Gamma_{C\bar{C}A}^R(1;2|\bar{3})D_{A^2}^R(\bar{3};\bar{2})\Gamma_{A^3}^R(\bar{2};3;4),
\end{aligned} \tag{3.8}$$

defining the renormalized Γ function

$$\Gamma^R_{C\bar{C}A}(1;2|3) = Z_1^{C\bar{C}A}\cdot\Gamma^{(0)}_{C\bar{C}A}(1;2|3)-\frac{g^2}{i}Z_1^{C\bar{C}A}\cdot\Gamma^{(0)}_{C\bar{C}A}(1;\bar{2}|\bar{3})$$
$$\times\, G^R_{C\bar{C}}(\bar{2};\bar{1})W_{C\bar{C}A^2}(\bar{1};2|3;4)D^R_{A^2}(4;\bar{3}). \quad (3.9)$$

We introduce a new four-point function $P_{C\bar{C}A^2}(1;2|3;4)$,

$$P_{C\bar{C}A^2}(1;2|3;4)$$
$$= W_{C\bar{C}A^2}(1;2|3;4)+\frac{g^2}{i}W_{C\bar{C}A^2}(1;\bar{2}|\bar{3};4)D^R_{A^2}(\bar{4};\bar{3})$$
$$\times\, G^R_{C\bar{C}}(\bar{2};\bar{1})P_{C\bar{C}A^2}(\bar{1};2|3;\bar{4}) \quad (3.10)$$

having used the block $W(1;2|3;4)$ as an irreducible part. Equation (3.8) defining the Γ-function is now equivalent to the following equation

$$\Gamma^R_{C\bar{C}A}(1;2|3) = Z_1^{C\bar{C}A}\cdot\Gamma^{(0)}_{C\bar{C}A}(1;2|3)-\frac{g^2}{i}\Gamma^R_{C\bar{C}A}(1;\bar{2}|\bar{3})G^R_{C\bar{C}}(\bar{2};\bar{1})$$
$$\times\, P_{C\bar{C}A^2}(\bar{1};2|3;\bar{4})D^R_{A^2}(\bar{4};\bar{3}) \quad (3.11)$$

which expresses $Z_1^{C\bar{C}A}\cdot\Gamma^{(0)}_{C\bar{C}A}$ only in terms of renormalized quantities. This is just the equation to be used for eliminating $Z_1^{C\bar{C}A}\cdot\Gamma^{(0)}_{C\bar{C}A}$ from all the other equations of the theory under consideration.

The combinations $Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}$ and $Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}$ are eliminated in an analogous way. It is convenient, however, to unite the equations for the corresponding vertex functions in the single matrix equation

$$[\Gamma^R_{A^3}(1;2;3)\quad \Gamma^R_{A^4}(1;2;3;4)]$$
$$= [Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}(1;2;3)\quad Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}(1;2;3;4)]$$
$$-[Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}(1;\bar{2};\bar{3})\quad Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}(1;\bar{2};\bar{3};\bar{4})]$$
$$\times\begin{bmatrix}\dfrac{\delta K_{A^2A}(\bar{2};\bar{3}|2)}{(ig)\delta\langle A^R(3)\rangle} & \dfrac{\delta^2 K_{A^2A}(\bar{2};\bar{3}|2)}{(ig)^2\delta\langle A^R(3)\rangle\delta\langle A^R(4)\rangle}\\ \dfrac{\delta K_{A^3A}(\bar{2};\bar{3};\bar{4}|2)}{(ig)\delta\langle A^R(3)\rangle} & \dfrac{\delta^2 K_{A^3A}(\bar{2};\bar{3};\bar{4}|2)}{(ig)^2\delta\langle A^R(3)\rangle\delta\langle A^R(4)\rangle}\end{bmatrix}$$
$$+\begin{bmatrix}\dfrac{\delta M'_{A^2}(1;2)}{(ig)\delta\langle A^R(3)\rangle} & \dfrac{\delta^2 M'_{A^2}(1;2)}{(ig)^2\delta\langle A^R(3)\rangle\delta\langle A^R(4)\rangle}\end{bmatrix} \quad (3.12)$$

because this simplifies considerably the consequent calculations. The new K-functions in (3.12) are directly connected with the self-energy operators introduced above:

$$\Pi'_{A^2}(1;\bar{1}) = -Z_1^{A^3}\Gamma^{(0)}_{A^3}(1;2;3)K_{A^2A}(3;2|\bar{1})$$
$$-Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}(1;2;3;4)K_{A^3A}(4;3;2|\bar{1}) + M'_{A^2}(1;\bar{1}). \quad (3.13)$$

the function F is defined by the corresponding matrix from equation (3.12)

$$\begin{bmatrix} F_{A^2A^2}(\bar{2};\bar{3}|2;3) & F_{A^2A^3}(\bar{2};\bar{3}|2;3;4) \\ F_{A^3A^2}(\bar{2};\bar{3};\bar{4}|2;3) & F_{A^3A^3}(\bar{2};\bar{3};\bar{4}|2;3;4) \end{bmatrix} = \begin{bmatrix} \dfrac{\delta K_{A^2A}(\bar{2};\bar{3}|2)}{(ig)\delta\langle A^R(3)\rangle} & \dfrac{\delta^2 K_{A^2A}(\bar{2};\bar{3}|2)}{(ig)^2\delta\langle A^R(3)\rangle\delta\langle A^R(4)\rangle} \\ \dfrac{\delta K_{A^3A}(\bar{2};\bar{3};\bar{4}|2)}{(ig)\delta\langle A^R(3)\rangle} & \dfrac{\delta^2 K_{A^3A}(\bar{2};\bar{3};\bar{4}|2)}{(ig)^2\delta\langle A^R(3)\rangle\delta\langle A^R(4)\rangle} \end{bmatrix}$$
$$+ \begin{bmatrix} \dfrac{\delta K_{A^2A}(\bar{2};\bar{3}|\bar{\bar{2}})}{(ig)\delta\langle A^R(\bar{\bar{3}})\rangle} & \dfrac{\delta^2 K_{A^2A}(\bar{2};\bar{3}|\bar{\bar{2}})}{(ig)^2\delta\langle A^R(\bar{\bar{3}})\rangle\delta\langle A^R(\bar{\bar{4}})\rangle} \\ \dfrac{\delta K_{A^3A}(\bar{2};\bar{3};\bar{4}|\bar{\bar{2}})}{(ig)\delta\langle A^R(\bar{\bar{3}})\rangle} & \dfrac{\delta^2 K_{A^3A}(\bar{2};\bar{3};\bar{4}|\bar{\bar{2}})}{(ig)^2\delta\langle A^R(\bar{\bar{3}})\rangle\delta\langle A^R(\bar{\bar{4}})\rangle} \end{bmatrix} \times \begin{bmatrix} F_{A^2A^2}(\bar{\bar{2}};\bar{\bar{3}}|2;3) & F_{A^2A^3}(\bar{\bar{2}};\bar{\bar{3}}|2;3;4) \\ F_{A^3A^2}(\bar{\bar{2}};\bar{\bar{3}};\bar{\bar{4}}|2;3) & F_{A^3A^3}(\bar{\bar{2}};\bar{\bar{3}};\bar{\bar{4}}|2;3;4) \end{bmatrix} \tag{3.14}$$

and makes it possible, some easy transformations being carried out, to obtain a simple equation for eliminating $Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}$ and $Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}$ combinations from all the consequent calculations:

$$[\Gamma^R_{A^3}(1;2;3) \quad \Gamma^R_{A^4}(1;2;3;4)]$$
$$= [Z_1^{A^3}\cdot\Gamma^{(0)}_{A^3}(1;2;3) \quad Z_1^{A^4}\cdot\Gamma^{(0)}_{A^4}(1;2;3;4)]$$
$$-[\Gamma^R_{A^3}(1;\bar{2};\bar{3}) \quad \Gamma^R_{A^4}(1;\bar{2};\bar{3};\bar{4})]$$
$$\times \begin{bmatrix} F_{A^2A^2}(\bar{2};\bar{3}|2;3) & F_{A^2A^3}(\bar{2};\bar{3}|2;3;4) \\ F_{A^3A^2}(\bar{2};\bar{3};\bar{4}|2;3) & F_{A^3A^3}(\bar{2};\bar{3};\bar{4}|2;3;4) \end{bmatrix} \tag{3.15}$$
$$+\begin{bmatrix} \dfrac{\delta M'_{A^2}(1;\bar{2})}{(ig)\delta\langle A^R(\bar{3})\rangle} & \dfrac{\delta^2 M'_{A^2}(1;\bar{2})}{(ig)^2\delta\langle A^R(\bar{3})\rangle\delta\langle A^R(\bar{4})\rangle} \end{bmatrix}$$
$$\times \begin{bmatrix} \delta(2;\bar{2})\delta(3;\bar{3}) + F_{A^2A^2}(\bar{2};\bar{3}|2;3) & F_{A^2A^3}(\bar{2};\bar{3}|2;3;4) \\ F_{A^3A^2}(\bar{2};\bar{3};\bar{4}|2;3) & \begin{matrix}\delta(2;\bar{2})\delta(3;\bar{3})\delta(4;\bar{4}) \\ +F_{A^3A^3}(\bar{2};\bar{3};\bar{4}|2;3;4)\end{matrix} \end{bmatrix}.$$

Now the renormalization programme may be considered to be over. The 'overlapping divergencies' being excluded with the aid of (3.11) and (3.15), the self-energy operators (3.6) can now be subject for explicit calculations. The factors Z_1 are eliminated from (3.11) and (3.15) by usual methods (Fradkin 1954, 1955b, 1965).

4. Asymptotic behaviour of the Green and vertex functions in the region of large transferred momentum

Asymptotic behaviour of the Green and vertex functions will be calculated by us within the above set of exact renormalized equations. We shall restrict ourselves to the so called 'three-gamma' approximation and carry out all calculation within it following the method of one of the present authors (Fradkin 1955a, 1955c). 'Three-gamma' approximation used by us here implies the simplification of the exact set of dynamical equations with the help of self-consistent expression of the highest vertices (or highest Green functions) through the exact one-particle Green functions and the simplest exact vertices.

Below we shall consider the equation for the vertex function of the ghosts only:

$$\Gamma_{C\bar{C}A}(1;2|3) = Z_1^{C\bar{C}A} \cdot \Gamma^{(0)}_{C\bar{C}A}(1;2|3) - \frac{g^2}{\mathrm{i}} \Gamma^R_{C\bar{C}A}(1;\bar{2}|\bar{3}) G^R_{C\bar{C}}(\bar{2};\bar{1}) P_{C\bar{C}A^2}(\bar{1};2|3;\bar{4}) D^R_{A^2}(\bar{4};\bar{3}) \tag{4.1}$$

and the set of equations for the derivatives of the corresponding Green functions:

$$\frac{\mathrm{d}[D^R_{A^2}]^{-1}(p^2)}{\mathrm{d}p^2} = 1 - \frac{\mathrm{d}\Pi^R_{A^2}(p^2)}{\mathrm{d}p^2}; \qquad \frac{\mathrm{d}[G^R_{C\bar{C}}]^{-1}(p^2)}{\mathrm{d}p^2} = 1 - \frac{\mathrm{d}\Sigma^R_{C\bar{C}}(p^2)}{\mathrm{d}p^2}. \tag{4.2}$$

We shall not need an equation for the Γ_{A^3}-function since the latter can be determined with the aid of the exact Ward identities; the Γ_{A^4}-function appears not to be connected with these calculations because the corresponding part of the self-energy operator does not contribute to the asymptotic region. The set of equations for the derivatives of the corresponding Green functions is as exact as the initial set of equations for the Green functions, but enables us to avoid some difficulties in calculations.

The simplifications of the exact set of equations (4.1), (4.2) within the 'three-gamma' approximation are first of all due to an approximate change of $Z_1^{C\bar{C}A} \cdot \Gamma^{(0)}_{C\bar{C}A}$ and $Z_1^{A^3} \cdot \Gamma^{(0)}_{A^3}$ in (4.1) and (4.2). We shall restrict ourselves here to the leading terms of equations (3.11) and (3.15) obtained above:

$$Z_1^{C\bar{C}A} \cdot \Gamma^{(0)}_{C\bar{C}A} \to \Gamma^R_{C\bar{C}A}, \qquad Z_1^{A^3} \cdot \Gamma^{(0)}_{A^3} \to \Gamma^R_{A^3}. \tag{4.3}$$

Besides, when solving equation (3.10) for the four-point function $P_{C\bar{C}A^2}$ one should also restrict oneself to the leading terms and simplify the expression for $W_{C\bar{C}A^2}$ within the same accuracy. As a result, instead of the exact equation (4.1) the vertex function of ghosts is to be determined from a still simpler equation,

$$\begin{aligned}\Gamma^R_{C\bar{C}A}(1;2|3) &= Z_1^{C\bar{C}A} \cdot \Gamma^{(0)}_{C\bar{C}A}(1;2|3) \\ &\quad - \frac{g^2}{\mathrm{i}} \Gamma^R_{C\bar{C}A}(1;\bar{2}|\bar{3}) G^R_{C\bar{C}}(\bar{2};\bar{1}) \Gamma^R_{C\bar{C}A}(\bar{1};2|\bar{\bar{3}}) D^R_{A^2}(\bar{\bar{3}};\bar{\bar{2}}) \Gamma^R_{A^3}(\bar{\bar{2}};3;4) D^R_{A^2}(4;\bar{3}) \\ &\quad - \frac{g^2}{\mathrm{i}} \Gamma^R_{C\bar{C}A}(1;\bar{2}|\bar{3}) G^R_{C\bar{C}}(\bar{2};\bar{1}) \Gamma^R_{C\bar{C}A}(\bar{1};\bar{\bar{2}}|4) G^R_{C\bar{C}}(\bar{\bar{2}};\bar{\bar{1}}) \Gamma^R_{C\bar{C}A}(\bar{\bar{1}};2|3) D^R_{A^2}(4;\bar{3}),\end{aligned} \tag{4.4}$$

which, together with equations (4.2) after the corresponding replacement of $Z_1^{C\bar{C}A} \cdot \Gamma^{(0)}_{C\bar{C}A}$ and $Z_1^{A^3} \cdot \Gamma^{(0)}_{A^3}$ combinations in it according to (4.3), is the basic for our further calculations. It should be noted here that the 'three-gamma' set of equations thus obtained may be derived in diagram language also. It is necessary in this case to replace 'bare' vertices and zero one-particle Green functions by exact vertices and exact one-particle Green functions in irreducible diagrams of first-order perturbation theory for the vertex $\Gamma^R_{C\bar{C}A}$ and for the derivatives of the mass $\mathrm{d}\Sigma^R_{C\bar{C}}(p^2)/\mathrm{d}p^2$ and polarization $\mathrm{d}\Pi^R_{A^2}(p^2)/\mathrm{d}p^2$ operators.

Calculation of the asymptotic behaviour of the Green and vertex functions within such an approximate set of equations is carried out according to one of the present authors (Fradkin 1955a, 1955c) and in the most complete form can be found in one of our papers (Fradkin and Kalashnikov 1974). Here we shall present only the final result of these calculations:

$$\Gamma_{C\bar{C}A} = h_{C\bar{C}} = \Gamma; \qquad d_{A^2} = \Gamma^{10/3}; \qquad \Gamma_{A^3} = \Gamma^{-4/3};$$

$$\Gamma = \left(1 + C_2(G)\frac{11}{3}\,\frac{g^2}{16\pi^2}\xi\right)^{-3/22}. \tag{4.5}$$

Expressions (4.5) show the asymptotic behaviour of the Green and vertex functions in the region of large transferred momentum and are asymptotically exact.

As to the expression for the invariant charge $g^2(t) = g^2\Gamma^2_{A^3}d^3_{A^2}$, it can easily be found with the aid of (4.5) in the following form:

$$g^2(t) = \frac{g^2}{1 + C_2(G)\frac{11}{3}(g^2/16\pi^2)\ln(p^2/m^2)}, \tag{4.6}$$

which differs essentially from the corresponding expression in the Abelian theories by the sign in the denominator. From (4.6) it is seen that at high energies the effective interaction tends to zero and the theory becomes asymptotically free. The 'bare' charge coincides with the asymptotic value of the invariant charge at $p^2 \to \infty$ and also tends to zero. At the same time the 'zero-charge' difficulty is absent from this theory since according to (4.6) the renormalized charge remains constant.

5. Concluding remarks

We have obtained here an exact and completely renormalized set of equations for the Green functions in the Yang–Mills field theory. This set of equations contains no divergencies and uncertainties and requires no regularization for its solution neither in perturbation theory nor beyond its scope. The asymptotic behaviour of the Green and vertex functions is found in the region of large transferred momentum. The connection between the 'bare' and experimental charges for the given class of theories is discussed. All the calculations are performed in the so called 'three-gamma' approximation. However, since the theory is asymptotically free the results obtained here are exact. The result for the invariant charge within this approximation coincides with that in the 'one-loop' approximation of the renormalization group equations. Within the present approach, however, we obtain not only the ultraviolet asymptotic behaviour of the invariant charge but, at the same time, also the asymptotic behaviour of the Green and vertex functions in the same domain of the momentum transferred.

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